Spiraling Along the (n-dimensional) Clock-tower:
a tour of the layered dimensions of music
(MATH 3210 Report Format)

Executive Summary

In this research project, I approached the daunting task of seeking an efficient, consistent, and accessible method of quantifying musical objects such as chord progressions, scales, cycles, and intervals. This is obviously beyond the scope of one paper, so I focused the research into three approaches to two examples: a numerical analysis, a graphical representation, and a conceptual model founded in music theory. I used the row and column values from the Z12 (Z*11) multiplication table for one example and the values of the chord progression made famous in John Coltrane's tune “Giant Steps” for the second example. Using the 12 semitone per octave approach of equal-tempered tuning, I labeled the pitch classes according to the elements of Z12:

Z12 {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
P {C,C#, D,D#, E, F,F#, G,G#, A, A#, B } C...

This reduced the calculations to within one octave modulo-12. I assigned the value of octave as a different variable, so that the set patterns would be easier to visualize. After representing both my data sets as numbers, graphical patterns, and musical patterns, I began to analyze the patterns. One form of analysis I used was SOM (self organizing map). SOM is a neural network driven algorithm which maps vector values of n-attributes to a two dimensional grid. The closer vectors are mapped together, the more similar they are. Based on the layout of the map, sometimes conclusions can be drawn to the interrelations of the many attributes in the various vectors.

After careful and detailed analysis using all these different techniques, I have charted out many highly useful characteristics, trends, and symmetries in the data. My overarching goal is the find a way to reduce redundancies in the language of musical analysis so useful patterns and relationships within the sets can be used without arcane terminology. I believe this has been a success in this regard at least, because now that I have an understanding of the symmetries, and have the tools necessary to represent these different relationships not only in musical notation, but in graphs, diagrams, matrices, magic cubes, and various topological spaces. And, any educator knows, the more ways you have to approach a particular problem, the greater the perspective you have on the context of the problem, and the greater your chance of finding a solution. Also, when the information gleaned from patterns in one domain, can be translated to information in another domain, entire levels of connections can arise naturally in the new perspective. Using these methods, particularly the ability to write vectors as chords with attributes corresponding to values for the root of the chord, the third, the fifth, and the seventh and so forth, leads to deeper understanding of how each voice moves into the closest voice in the next chord (voice leading, a precious aspect of music often lost in more generalized music forms. In this form, however, it can be seen fairly clearly) This allowed me to approach the Coltrane Changes, which to most college level jazz students is a dreaded task due to its complexity and unusual chord choices, with clarity and almost ease. It is easy for the understanding of the inner movements of a chord progression to be lost in the parade of chord names and types. Using the matrix analysis and graphing the fundamental pattern within the matrix, I produced a diagram that shows the entire chord progression in any key based on the initial value of the root note. And within this drawing, are many possible teachings all tucked into its many intersections and folds that arose naturally from the stove when these musical patterns were put on to boil..

Problem Description

When approaching music without much background in musical theory, the seemingly endless lists of definitions, relationships, dots, and lines look, to the uninitiated, about as welcoming as an advanced college physics book to the beginning arithmetic student. Even if the physics book is explaining something as simple as a how to convert between different measuring systems, it may appear absolutely inaccessible without some context for the strange markings on the page. As a music teacher and a student of mathematics, I have in both areas experienced the daunting task of being familiar with the end product without knowing the path, method, or intention of its creator/creation. An example of this is listening to music for years without knowing 'what' they were playing, or in math, knowing the formulas and equations from science, but having no idea how they were derived. My goal as a teacher (one who is very passionate about the pursuit of knowledge, understanding, and applicable skills) is to find a way to teach potential students, who may become great at math, music, art, or any discipline but who are ultimately prevented from this due to lack of understanding of the material at hand. There are many children and adults alike, of all ages who want nothing more than to be able to express themselves on an instrument, but feel they simple don't have the “talent” or the time to learn how to. The problem is almost always related to the gap that tends to form between the available information and their understanding of the institutionalized systems which are firmly in place (and I am not suggesting they be moved) This happens in a different way with math, but to the same end result. They are told, or they come to the conclusion that they don't have a “mathematical mind” or that they aren't smart enough to learn math. This is almost never the case. What tends to happen is that the millions of different minds contained within our extremely diverse country, are all funneled through the exact same education system, with hard-lined, preconceived and rigidly defined notions about how to teach, qualify, and relocate students throughout the various stages of the education system. Understanding of how all these different areas interrelate and, perhaps even more importantly, how they can be applied is often lost in the parade of compartmentalized subjects, tests, and rigid expectations. This is particularly evident when it comes to the imposed dichotomy between the arts and the sciences, or between creative disciplines and more quantitative ones. It is common for kids to be taught or to assume that they can't be both good at science and music, or at least that it is very rare and difficult; but when one looks historically into the innovators of math, science, and music, these were people who understood many different areas of the mind. They didn't operate exclusively in the realm of one discipline or the other, and often they were so closely bound, they overlapped. A few crucial examples: Pythagoras, Plank, and Einstein. All these men were not only powerful in the areas of math and science, they were also either avid players or theoretic developers.

My goal, on a much larger scale than this particular project, is to constantly teach integrative disciplines to all my students. I will use math to teach music, music to show students they can learn math, and many other pairings of normally separated disciplines in order to always bring to bear ones greater understanding of all that has been learned so that solutions normally hidden just behind a nearby curtain may be used to solve problems in different domains.

Now, in this particular problem, due to the enormous problem domain of defining every possible relationship in music in a systematic way, I have focused the scope to discrete sets of pitches, and have spent the majority of my analytical energies attempting to seek out equivalences within the music that will lead to a refined notation that will greatly reduce redundancies and will allow the core content of some very fundamental musical relationships shine through. The goal is to develop/refine a system of musical analysis that consistently, efficiently, and in a translatable manner (between systems), quantifies musical objects. I have used various techniques (explained in detail below) such as modular arithmetic, basic graph theory, discrete algebraic structures, set theory, linear algebra, computer analysis (spreadsheets, C++, and data mining algorithms to name a few), advanced papers on music theory (written by people better classified as mathematicians), basic topology, and my own system of drawing cycle diagrams to represent intra-octave relationships. A question I have let guide me is this:

Assuming there are symmetries in both the sound, the structures, and in the theoretical representation of music, how can we go about quantifying, enumerating, and relating them in order to reduce redundancies, reveal fundamental patterns, and make accessible the arcane hallways of western musical theoretical language?

In one example I examine a set of cycles which partition the Z12 set (which when combined produce the K12 complete graph with no line drawn twice) and in the other I use similar methods to visualize the patterns within the John Coltrane song, “Giant Steps”.

Analysis Technique

I have approached each set of data on at least these three levels: numerical, graphical, and theoretical (in terms of western music theory). I have made the organization and preparation of all the data as the primary desired result of this exercise. If done in a consistent manner, even massive data bases of musical (or just numerical) data can be prepared in a way that searching, clustering, and decision algorithms, along with a variety of other statistical and data mining techniques, may be used to discover knowledge in the sets. (statistical analysis != data mining, a common, frustrating, and often misleading misconception) The first most important feature is the numerical set which represents all twelve tones within an octave (see appendix B). This allows one consider the value of the octave as a separate piece of information and to assign the octave value as a scalar. This is helpful because this allows us to reduce the many pitches spanning however many octaves the instrument has to the range of one octave. There will be times when spanning multiple octaves and knowing the exact distances between the points becomes very important, and for more musically-specific forms of analysis, this can be approached with various subscript notations. In most aspects of the current research we will not need this to use this, but with the scalar values and subscript notation, the precise placement of the note in terms of both pitch class and octave can easily be calculated. Considering that the average maximum range of human hearing perception goes from 20HZ to 20,000HZ (20kHZ)[1] and knowing that an octave is a doubling of a pitch frequency, all we need to do to calculate the total range in which we might need to analyze the data, is to find out how many times 20 can be double before it exceeds 20,000. It turns out that this number is a little less than ten. This means that the largest octave range we would need to explore for applicable music, is a little less than 10. To visualize this (we will come back to this technique again) imagine a clock. It has twelve numbers, but instead of 12, we will call it zero (the fact that it is not called zero stems from the lag in the west of the concept of zero. Unfortunately, this topic is out of the range of out current pursuits.) A pitch class, is analogous to one of the of the twelve places on the clock. The analogous octave concept is to imagine that there is not one clock, but layers of clocks rising above and below. The value of octave is which level of the “clock tower” the pitch is on, and the class of pitch is the orientation along the twelve spots on the ring. Any of the diagrams contained within this project have more than three dimensions but due to spacial limitations, have been reduced to two and three dimensional diagrams.

The data sets I used were numerical representations of on certain pattern, or at the very least prepared. In the first example I used the Z12 multiplication table (see appendix A below or chart included), entered it first into a text file, then into a spread sheet, and considered the rows and columns as vectors. The columns and the rows are identical due to the symmetry about the diagonal, so either could serve as input vectors for the various forms of analysis. In the more specific example, I narrowed the focus down to a finer level and analyzed the chord vectors of the John Coltrane tune “Giant Steps”. I picked these two examples to be used as the subjects for all three forms of analysis, because within the Z12 multiplication (it can also be represented as Z*11) table is contained a great deal of information about the various intervals between pitches, inversions of patterns, transposing patterns/key changes, scale and chord patterns, distances, and voice leading common tones. All of this is shown in a general form, by this I mean that the

1 http://en.wikipedia.org/wiki/Hearing_range

patterns will be based off of a choice of the pitch which will be called 0. This allows patterns to be formed in relationship to themselves and other patterns, and to then easily be shifted through the twelve possible root tones. Combining the pitch cycle, the octaves, the choice of root note, the relationship of these patterns with respect to time (rhythm), as well as considering the number of voices independently having varying amounts of all the above attributes, the number of dimensions ranks higher than three. How many more than three? It depends on how many instruments and voices we have interacting in time and how many of these voices are independent as opposed to scalar multiples each other. We will be thinking of dimension in a more generalized sense and think of it as the minimum number of coordinates needed to specify every point within the given system.

This point leads to the core of this research. This project has been a data mining project in many ways. First of all, I did use a data mining algorithm to analyze the data at a certain point in the project. However, it is very important for me to note, the information I mined from these data sets was not only what the algorithm showed, but more so the understanding gained of how to represent musical objects with various values, how to manipulate and visualize them in a spreadsheet, how to represent them graphically, and how to create analogies that paralleled some of the core features.

Z12 Analysis

The first step in this analysis was to make the Z12 multiplication table. I considered my set to be S= {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} and any values I got 12 or higher, I divided by twelve, took the remainder, and that value is the pitch class reduced back to one octave. I used a text editor and then a spreadsheet to create a few of these tables. I spent some time with pencil, paper, compass, and ruler exploring each of the columns and rows of the object. I noted that along the sixth row and sixth column, the values of the rows and columns invert. The values in the row for five, are listed in reverse order in the column for 7. The same is true for 4 and 8, 3 and 9, 2 and 10, 1 and 11. At this point I used the diagram of the “circle of fifths” the model used to represent the values and order of the progression through the twelve key signatures by the movement of the perfect fifth (an interval equivalent to 7 semitones) when traversed clockwise, and by perfect fourth (five semitones) when traversed counter-clockwise. Normally, when this diagram is drawn, it is in terms of pitches as letters, and keys with sharps and flats. When looked at this way, it is very difficult to see consistent patterns in it. However, when rewritten with numbers instead of letters, and calling the C at 12o'clock on the circle as 0, many patterns become obvious. These patterns by having a direct correlation to musical entities, have musical meaning and can be used to develop visual aids to see larger scale patterns. I used two ways of labeling the diagram: one in the order of the clock, and another in the order of the circle of fifths. Using these and the data from the Z12 columns, I observed that the if the values of the columns were drawn as pathways onto the circle, once all six classes where drawn onto the circle, every point was connected to every other point exactly once, meaning that the set of intervals produced six equivalence classes which completely partitioned the set of pitches. When complete, the end product of this map is the K12 complete graph. From these drawings, I was able to make many observations about symmetries within the data (see results below). After looking at the table itself, then the graphical representation, I tried to also conform the data to a form that it could become input for the SOM (Aleshunas). I converted the rows of the table into 11 classes of 12 attribute vectors which served as the input to the self organizing map.

SOM

The self organizing map is a data mining utility used to cluster and relate n-dimensional input vectors, which it then maps each vector to a square on a n x m grid based on similarities between the vectors such as inner product value or frequency of occurrence. The version we used (Aleshunas) required us to use batch files with information about the number of attributes, the dimension of the grid the vectors are being mapped to, the number of iterations of the training algorithm for both a rough mapping and a highly refined mapping, the value of error correction applied back to the map, and the labels of each class. The SOM is an unsupervised Neural Network based algorithm which compares vectors of higher dimension and maps them onto a 2-dimensional plane. This allows us to visualize relationships between vectors of higher order.

I applied the SOM to cluster both Z12 (eleven 12 attribute vectors) and the set of chord vectors from the Coltrane Matrix (six 4 attribute vectors). For the Z12 example, after submitting the vectors in several different attempts, I eventually came out with a 3x4 map, which clustered the values in a manner described below. After completing all the different forms of analysis on the Z12 members thus far, I then looked at the data I had gathered, mentally and with many drawings, converted the information back into musical terms, and submitted the ideas to the lens of music theory. This added yet another dimension to the understanding of the relationships within the data. After finishing this analysis and recording the various results, I prepared some spreadsheet graphs which confirmed visually some of the observations made before hand in the previous methods. Next, I repeated this process of music to number, number to relationships, relationships to graphs, all this to the SOM, and then a final run through to compare the information to know aspects of music. I then repeated this process on the Coltrane input data. The only difference between the two was the manner of input. In the Coltrane example, there were sets of six, four-attribute vectors, with the attributes corresponding to the root of the chord with respect to the progression, the third away from that, the fifth, and then the seventh. This example was a more specific set, with less attributes in its vectors, so it mapped nicely illustrating some of the points arrived at in the other methods.

Assumptions

The assumptions made are very important when it comes to cross method analysis. When using multiple languages such as music in math, sometimes the terms and seemingly obvious parts don't always correlate to reflective cases where it seems the same sort of symmetry would hold. For instance, the tritone, the value of 6 in the Z12 system seems to be very important based on its number patterns alone. It is the value that cuts the system in half and has a very nice evenly divisible value with 12 the size of the system, yet, the tritone is so dissonant that during more dogmatic eras of music, if one played this interval before a high ruler, or at an important public function, they could be put in jail. It was referred to as “diablos en musica” the devil of music. This doesn't change the importance of the tritone as discussed above, but it is important to realize that depending on the musical context, patterns that seem highly symmetrical, can at times have effects that are less than desirable. Another important assumption is the assumption of equal tempered tuning which artificially divides the octave into twelve equal parts for the sake of discrete instruments and for the sake of having the sensation of changing keys. This causes some harmonic problems because even though the circle of fifths suggests that twelve fifths (seven semitones) adds up to seven octaves (twelve semitones). According to semitones, they should both add up to an even eighty-four semitones, where actually the twelve fifths comes just a little short. In turn, the piano, and western tuning in general, assumes an equal amount of out of tune-ness for every note. To confront this, the more advanced system that includes real values and utilizes topographical methods of higher dimensions would be needed to include all types of tuning as well as all the subclasses within each one.

Results

Cycle diagrams, charts from spreadsheet, Giant Steps matrix, vector values between every other chord, SOM maps, and tables of inversions. Results also include the new model of visualization granted from the tower of cycles.

Cycles in music are pathways taken through an octave that lead in one direction. So, the chromatic scale is the most inclusive of cycles because it leads by semitone from a starting tone step-wise through each element until each reaches the next octave (above or below) which will be the same modular value as the starting tone.

Here are the results of the cycle classes I derived:

Cycle classes:
{0}-(0, 12, 24, ...)   //same as 12
{1}-(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
{2}-(0, 2, 4, 6, 10)
{3}-(0, 3, 6, 9)
{4}-(0, 4, 8)
{5}-(5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7)
{6}-(0, 6)
{7}= inverse of {5}
.
.
.
ex. {7}=(7 2 9 4 11 6 1 4 3 2 7)
which is clearly {5} in reverse order.

All of the cycle patterns can be stretched through the different keys resulting in these patterns times twelve instantiations.

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*The only cycle that doesn't contain the element 6, is the 4/8 cycle.
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Inverses:
from some pitch p,
p+x=p+(-x)

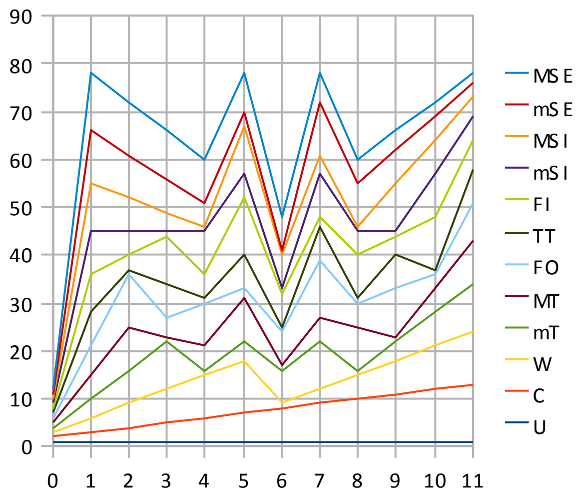
Using pitch 6:

-x	x	+x
6	0	6
5	1	7
4	2	8
3	3	9
2	4	10
1	5	11
0	6	0

There exists in this system a reflective symmetry about the element 6.

Unique-
moving by +/-5 or by +/-1 are the only two sets that generate the entire set without repeating. The two exceptions to this are the 7 and 11 sets; however, they are not actually different because in a mod12 system, 5 and 7 are negatives as are 1 and 11.

Here is a graph showing the symmetry about the tritone (6) made from SOM output:



The kink in the middle of the graph shows the symmetry about the sixth element, the tritone.

Here is the Coltrane Matrix as derived in a spreadsheet:

Measure	1	2	3	4
1	Bmaj7 D7	Gmaj7 Bb7	Ebmaj7 -	Am7 D7
	0 3	8 11	4 -	10 3
5	Gmaj7 Bb7	Ebmaj7 F#7	Bbmaj7 -	Fm7 Bb7
	8 11	4 7	0 -	6 11
9	Ebmaj7 -	Am7 D7	Gmaj7 -	C#7 F#7
	4 -	10 3	8 -	2 7
13	Bmaj7 -	Fm7 Bb7	Ebmaj7 -	C#m7 F#7
	0 -	6 11	4 -	2 7

Each square is a measure of 4/4 time, and the numbers represent the root of the chord in the measure. Here is the Coltrane changes in general, in every key which I derived by transposition within a spreadsheet:

```
0 3 8 11 4 7 0
4 7 0 3 8 11 4
7 10 3 6 11 2 7
11 1 7 9 4 5 11

1 4 9 0 5 8 1
5 8 1 4 9 0 5
8 11 4 7 0 3 8
0 2 8 10 5 6 12

2 5 10 1 6 9 2
6 9 2 5 10 1 6
9 0 5 8 1 4 9
1 3 9 11 6 7 1
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4 7 0 3 8 11 4
8 11 4 7 0 3 8
11 2 7 10 3 6 11
3 5 11 1 8 9 3

6 9 2 5 10 1 6
10 1 6 9 2 5 10
1 4 9 0 5 8 1
5 7 1 3 10 11 5

8 11 4 7 0 3 8
0 3 8 11 4 7 0
3 6 11 2 7 10 3
7 9 3 5 0 1 7

10 1 6 9 2 5 10
2 5 10 1 6 9 2
5 8 1 4 9 0 5
9 11 5 7 2 3 9

1 4 9 0 5 8 1
5 8 1 4 9 0 5
8 11 4 7 0 3 8
0 2 8 10 5 6 12

3 6 11 2 7 10 3
7 10 3 6 11 2 7
10 1 6 9 2 5 10
2 4 10 0 7 8 2

5 8 1 4 9 0 5
9 0 5 8 1 4 9
0 3 8 11 4 7 0
4 6 0 2 9 10 4

7 10 3 6 11 2 7
11 2 7 10 3 6 11
2 5 10 1 6 9 2
6 8 2 4 11 0 6

9 0 5 8 1 4 9
1 4 9 0 5 8 1
4 7 0 3 8 11 4
8 10 4 6 1 2 8

11 2 7 10 3 6 11
3 6 11 2 7 10 3
6 9 2 5 10 1 6
10 0 6 8 3 4 10

Each four note column represents a chord in the series, and each set of seven chords is the whole set of changes in that key.

SOM mapping results for Coltrane:

eight	X	X	X
X	R	X	three
eleven	X	four	three
eleven	seven	X	four
X	seven	three	four
X	X	three	three
X	X	four	three

Each word represents the number of the root chord in the chosen key. Every chord is mapped next to the chords I found them to be attached to in the graphical representation. SOM was able to determine the relationship that existed between the chord vectors and map them those which were most related near each other.

Issues

Many of the Issues have been discussed in the assumption section, but a final point to make about these methods is that there has always been, sense the implementation of equal temper tuning and before, a conflict between the desire to have low ration harmonic frequencies, and to have evenly spaced numbers. In this project, I chose to work with the latter, but to include the other approach can be achieved by transforming this system into a Real number based system instead of a discrete one. I am currently working on a way to transform this system into this level of complexity. Another issue I had was with the SOM mapper. I could not get it to map out the vector slices, so I just took pieces of the data to demonstrate certain trends, proximities, and symmetries. The graphs included in the spreadsheet format are from the SOM output.

Appendices

A: Z12 table (zero excluded, making this Z*11)

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	1	2	3	4	5	6	7	8	9	10	11
2	2	4	6	8	10	0	2	4	6	8	10
3	3	6	9	0	3	6	9	0	3	6	9
4	4	8	0	4	8	0	4	8	0	4	8
5	5	10	3	8	1	6	11	4	9	2	7
6	6	0	6	0	6	0	6	0	6	0	6
7	7	2	9	4	11	6	1	8	3	10	5
8	8	4	0	8	4	0	8	4	0	8	4
9	9	6	3	0	9	6	3	0	9	6	3
10	10	8	6	4	2	0	10	8	6	4	2
11	11	10	9	8	7	6	5	4	3	2	1

Set of numbers used to describe set of pitches:

S{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}
P{C,C#, D,D#, E, F,F#, G,G#, A, A#, B } C...

References

http://en.Wikipedia.org/wiki/
-Hearing_range
-Dimension
-12_tone
-Dihedral_group
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Aleshunas, JJ; SOM